Tutorial 12. 30-11-2016

Topic: Isolated zeros and revision exercise.

Thm: If f is analytic on Ω and $f \not\equiv 0$ on Ω, then the zeros of f are isolated.

Example: 1.) Let f & g be analytic and $f(z), g(z) \neq 0$ on $D = \{z \mid |z| \leq 1\}$.

Assume that $\forall n \geq 2$, we have

$$\frac{f'(\frac{1}{n})}{f(\frac{1}{n})} = \frac{g'(\frac{1}{n})}{g(\frac{1}{n})}$$

Show that $\exists c \in \mathbb{C}$ s.t. $f(z) = c\,g(z)$ $\forall z \in D$.

Ans: Let $h(z) = \dfrac{f(z)}{g(z)}$

Note that $h(z)$ is analytic on D.

Moreover, $h'(z) = \dfrac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$

By assumption, we have

$$h'(\tfrac{1}{n}) = \frac{g(\frac{1}{n})f'(\frac{1}{n}) - f(\frac{1}{n})g'(\frac{1}{n})}{g(\frac{1}{n})^2}$$

$$= 0$$

Hence $h'(0) = \lim_{n \to \infty} h'(\frac{1}{n}) = 0$.

In particular, 0 is a zero of $h'(z)$ but it is not isolated, we have $h'(z) \equiv 0$ on D.

Hence $h(z) = c$ for some constant $c \in \mathbb{C}$.

2) Show that there does not exist an entire function satisfying $f(\frac{1}{n}) = e^{-n}$.

Ans: Since f is analytic and hence continuous,

$$f(0) = \lim_{n\to\infty} f(\tfrac{1}{n}) = \lim_{n\to\infty} e^{-n} = 0.$$

Thus $\exists\, m \in \mathbb{N}$ and an analytic function $g(z)$ s.t. $f(z) = z^m g(z)$ with $g(0) \neq 0$.

$$\Rightarrow \quad f(\tfrac{1}{n}) = \frac{g(\frac{1}{n})}{n^m}$$

$$\Rightarrow \quad g(\tfrac{1}{n}) = \frac{n^m}{e^n}$$

$$\Rightarrow \quad g(0) = \lim_{n\to\infty} g(\tfrac{1}{n}) = \lim_{n\to\infty} \frac{n^m}{e^n} = 0, \text{ contradiction.}$$

So no such f exists.

3) (MATH2230 A/B, 2014-2015, Q3)
Suppose f is an analytic function on $\Omega = \{z \mid |z| \in (0,1)\}$.
Moreover assume f satisfies

$$|f(z)| \le \log\left(\frac{1}{|z|}\right) \quad \forall z \in \Omega$$

Show that f can be extended to the disk $\{z \mid |z| \le 1\}$ and $f \equiv 0$ on the disk.

Ans: Since f is analytic on Ω,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-0)^n,$$

where $$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}}\, dz, \quad R \in (0,1).$$

In particular,

$$|a_n| \leq \frac{1}{2\pi}(2\pi R)\frac{\log\left(\frac{1}{R}\right)}{R^{n+1}} \to 0 \quad \text{as } R \to 1$$

Hence we have $a_n = 0 \quad \forall n \in \mathbb{Z}$.

$$\Rightarrow \quad f(z) = 0 \quad \forall z \in \Omega.$$

In particular f can be extended to the whole disk.